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Boundary spontaneous polarization in the six-vertex model with a reflecting boundary

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Abstract

We consider the six-vertex model on a $2N \times N$ lattice with domain wall boundary conditions for the lower, upper and right boundaries and a reflecting end for the left boundary. The boundary one-point correlation functions, which describe the boundary spontaneous polarization, are calculated and expressed as some determinants of $N \times N$ matrices.

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1. Introduction

The six-vertex model is an important model of statistical mechanics in two-dimensional space. It was first introduced in [1] as an ice model, and was solved exactly [2, 3]. Later, the six-vertex model was studied extensively not only with periodic boundary conditions, but also with different boundary conditions: for example, the domain wall boundary [4] and the reflecting boundary [5]. With these different boundary conditions, the six-vertex model was proved to be solved exactly, and its partition functions allow us to derive determinant representations [6–8]. Its boundary polarization correlation functions were also solved exactly [9–11]. In this paper, we discuss the six-vertex model with reflecting boundary conditions. We calculate four types of correlation functions, which are related to the boundary polarization, and we derive the determinant representations for these correlation functions.

2. Six-vertex model with a reflecting boundary condition

In this section, we introduce the model and some notations. We discuss a $2N \times N$ lattice as shown in figure 1(a).

There are three convenient ways to describe the six-vertex model, which are in terms of arrows, lines and spins [9]. In this paper, the spin description is adopted. The boundary

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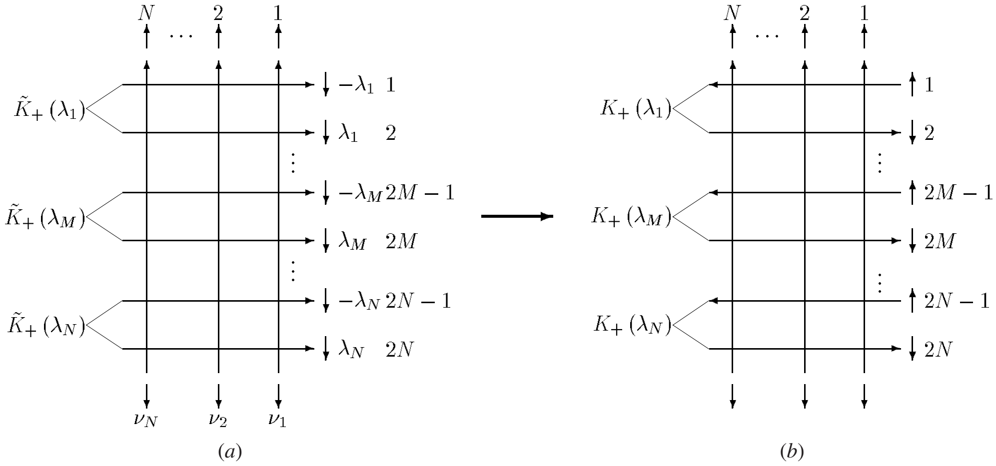

Figure 1. A six-vertex model with reflecting boundary, in terms of arrows.

conditions are the domain wall boundary for the lower, upper and right boundaries and a reflecting end for the left boundary. The reflecting matrices on the left boundary are

$$\tilde{K}_+(\lambda_\alpha) = \begin{pmatrix} 0 & \cosh\left(\lambda_\alpha + \frac{1}{2}\eta - \frac{i\pi}{2} + \zeta_+\right) \\ \cosh\left(-\lambda_\alpha - \frac{1}{2}\eta + \frac{i\pi}{2} + \zeta_+\right) & 0 \end{pmatrix}$$

where ζ_+ is a boundary parameter. Vertical lines are enumerated by Latin indices ($k = 1, \ldots, N$) while horizontal lines are labelled with Greek indices $\alpha = 1, \ldots, 2N$). Each spin variable is situated on a lattice edge and takes two different values, $\uparrow$ or $\downarrow$. In the framework of the Bethe ansatz algebra, the statistical weight $L_{\alpha k}$ associated with the intersection of the αth row and kth column is defined as

$$L_{\alpha k}(\lambda_\alpha, \nu_k) = L_{\alpha k}(\lambda_\alpha - \nu_k) = \begin{pmatrix} \dfrac{\sinh\left(\lambda_\alpha - \nu_k + \frac{1}{2}\eta\sigma_k^3\right)}{\sinh\left(\lambda_\alpha - \nu_k + \frac{1}{2}\eta\right)} & \dfrac{\sinh(\eta)\sigma_k^-}{\sinh\left(\lambda_\alpha - \nu_k + \frac{1}{2}\eta\right)} \\ \dfrac{\sinh(\eta)\sigma_k^+}{\sinh\left(\lambda_\alpha - \nu_k + \frac{1}{2}\eta\right)} & \dfrac{\sinh\left(\lambda_\alpha - \nu_k - \frac{1}{2}\eta\sigma_k^3\right)}{\sinh\left(\lambda_\alpha - \nu_k + \frac{1}{2}\eta\right)} \end{pmatrix}_{[\alpha]}. \tag{1}$$

$\sigma_k^{1,2,3}$ are Pauli matrices acting on the kth column, $\sigma_k^\pm = \frac{1}{2}\left(\sigma_k^1 \pm i\sigma_k^2\right)$. To calculate the partition function, we consider the $(2\alpha - 1)$ and (2α) columns first, $\downarrow_{2\alpha} T^{t_M}(\lambda_M)\,\tilde{K}_+(\lambda_M)\,T(-\lambda_M) \downarrow_{2\alpha-1}$, where $T(\lambda_\alpha) = L_{\alpha N}(\lambda_\alpha, \nu_N)\cdots L_{\alpha 1}(\lambda_\alpha, \nu_1)$. The effect of these two columns is equal to

$$\downarrow_{2\alpha} U^{t_M}(\lambda_M) \uparrow_{2\alpha-1} = \mathcal{B}(\lambda_\alpha).$$

$U(\lambda_M)$ is the two-row monodromy matrix in the Bethe ansatz algebra [5] as shown in figure 1(b),

$$\begin{aligned} U^{t_\alpha}(\lambda_\alpha) &= T^{t_\alpha}(\lambda_\alpha) K_+(\lambda_\alpha)\sigma_\alpha^2 T(-\lambda_\alpha)\sigma_\alpha^2 = \begin{pmatrix} \mathcal{A}(\lambda_\alpha) & \mathcal{C}(\lambda_\alpha) \\ \mathcal{B}(\lambda_\alpha) & \mathcal{D}(\lambda_\alpha) \end{pmatrix} \\ K_+(\lambda_\alpha) &= \tilde{K}_+(\lambda_\alpha)\sigma_\alpha^2 = \begin{pmatrix} \sinh\left(\lambda_\alpha + \frac{1}{2}\eta + \zeta_+\right) & 0 \\ 0 & \sinh\left(-\lambda_\alpha - \frac{1}{2}\eta + \zeta_+\right) \end{pmatrix}. \end{aligned} \tag{2}$$

Then the partition function is equal to

$$Z_N(\{\lambda_\alpha\}_N, \{\nu_k\}_N) = w_N^- \prod_{\alpha=1}^N \mathcal{B}(\lambda_\alpha)\, w_N^+.$$

$w_N^{\pm}$ are pseudo-vacuum, $w_N^+ = \prod_{i=1}^N \uparrow_i$, $w_N^- = \prod_{i=1}^N \downarrow_i$. The monodromy matrices defined in equation (2) satisfy the boundary Yang–Baxter relation:

$$\begin{aligned} &R_{12}(-\lambda_1+\lambda_2)U^{t_1}(\lambda_1)R_{12}(-\lambda_1-\lambda_2-\eta)U^{t_2}(\lambda_2) \\ &\qquad = U^{t_2}(\lambda_2)R_{12}(-\lambda_1-\lambda_2-\eta)U^{t_1}(\lambda_1)R_{12}(-\lambda_1+\lambda_2). \end{aligned} \tag{3}$$

R_{12} is the trigonometric solution of the Yang–Baxter equation:

$$R_{12}(\lambda) = \begin{pmatrix} \sinh(\lambda+\eta) & & & \\ & \sinh(\lambda) & \sinh(\eta) & \\ & \sinh(\eta) & \sinh(\lambda) & \\ & & & \sinh(\lambda+\eta) \end{pmatrix}_{[12]}.$$

From equation (3), the commutation relations between $\mathcal{A}(\lambda_\alpha)$, $\mathcal{B}(\lambda_\alpha)$, $\mathcal{C}(\lambda_\alpha)$ and $\mathcal{D}(\lambda_\alpha)$ can be obtained. Some of these relations are

$$\begin{aligned} \mathcal{A}(\lambda_\alpha)\mathcal{B}(\lambda_\beta) &= \frac{\sinh^2(\lambda_\alpha-\eta)-\sinh^2\lambda_\beta}{\sinh^2\lambda_\alpha-\sinh^2\lambda_\beta}\mathcal{B}(\lambda_\beta)\mathcal{A}(\lambda_\alpha) + \frac{\sinh\eta\sinh(\lambda_\alpha+\lambda_\beta-\eta)}{\sinh^2\lambda_\alpha-\sinh^2\lambda_\beta}\mathcal{B}(\lambda_\alpha)\mathcal{A}(\lambda_\beta) \\ &\quad - \frac{\sinh(2\eta)\sinh(\eta)}{\sinh^2\lambda_\alpha-\sinh^2\lambda_\beta}\mathcal{B}(\lambda_\beta)\mathcal{D}(\lambda_\alpha) + \frac{\sinh\eta\sinh(2\eta-\lambda_\alpha+\lambda_\beta)}{\sinh^2\lambda_\alpha-\sinh^2\lambda_\beta}\mathcal{B}(\lambda_\alpha)\mathcal{D}(\lambda_\beta) \end{aligned} \tag{4}$$

$$\begin{aligned} \mathcal{D}(\lambda_\alpha)\mathcal{B}(\lambda_\beta) &= \frac{\sinh^2(\lambda_\alpha+\eta)-\sinh^2\lambda_\beta}{\sinh^2\lambda_\alpha-\sinh^2\lambda_\beta}\mathcal{B}(\lambda_\beta)\mathcal{D}(\lambda_\alpha) \\ &\quad - \frac{\sinh\eta\sinh(\lambda_\alpha+\lambda_\beta+\eta)}{\sinh^2\lambda_\alpha-\sinh^2\lambda_\beta}\mathcal{B}(\lambda_\alpha)\mathcal{D}(\lambda_\beta) + \frac{\sinh\eta}{\sinh(\lambda_\alpha+\lambda_\beta)}\mathcal{B}(\lambda_\alpha)\mathcal{A}(\lambda_\beta) \\ \mathcal{B}(\lambda_\alpha)\mathcal{B}(\lambda_\beta) &= \mathcal{B}(\lambda_\beta)\mathcal{B}(\lambda_\alpha). \end{aligned} \tag{5}$$

The eigenvalues of $\mathcal{A}(\lambda_\alpha)$ and $\mathcal{D}(\lambda_\alpha)$ when they act on w_N^+ are

$$\begin{aligned} &\mathcal{A}(\lambda_\alpha)w_N^+ = \Delta_+^N(\lambda_\alpha)w_N^+ \qquad \mathcal{D}(\lambda_\alpha)w_N^+ = \Delta_-^N(\lambda_\alpha)w_N^+ \\ &\Delta_+^N(\lambda_\alpha) = -\frac{\sinh(2\lambda_\alpha+\eta)}{\sinh(2\lambda_\alpha)}\delta_N(-\lambda_\alpha) + \frac{\sinh\eta}{\sinh(2\lambda_\alpha)}\delta_N(\lambda_\alpha) \\ &\Delta_-^N(\lambda_\alpha) = -\delta_N(\lambda_\alpha) \\ &\delta_N(\lambda_\alpha) = \sinh\left(\lambda_\alpha+\frac{1}{2}\eta-\zeta_+\right)\prod_{i=1}^N \frac{\sinh\left(\lambda_\alpha-\nu_i-\frac{1}{2}\eta\right)}{\sinh\left(\lambda_\alpha-\nu_i+\frac{1}{2}\eta\right)}. \end{aligned}$$

Now we define the following one-point correlation functions:

$$\begin{aligned} f_1^M &= Z_N^{-1}w_N^-\left[\prod_{\alpha=M+1}^N \mathcal{B}(\lambda_\alpha)\right]\left[\sinh\left(\lambda_M+\frac{1}{2}\eta+\zeta_+\right)q_1B(\lambda_M)p_1D(-\lambda_M)\right. \\ &\quad \left. + \sinh\left(\lambda_M+\frac{1}{2}\eta-\zeta_+\right)q_1D(\lambda_M)p_1B(-\lambda_M)\right]\left[\prod_{\alpha=1}^{M-1}\mathcal{B}(\lambda_\alpha)\right]w_N^+ \\ f_2^M &= Z_N^{-1}w_N^-\left[\prod_{\alpha=M+1}^N \mathcal{B}(\lambda_\alpha)\right]\left[\sinh\left(\lambda_M+\frac{1}{2}\eta+\zeta_+\right)B(\lambda_M)q_1D(-\lambda_M)p_1\right. \\ &\quad \left. + \sinh\left(\lambda_M+\frac{1}{2}\eta-\zeta_+\right)D(\lambda_M)q_1B(-\lambda_M)p_1\right]\left[\prod_{\alpha=1}^{M-1}\mathcal{B}(\lambda_\alpha)\right]w_N^+ \\ f_3^M &= Z_N^{-1}w_N^-\left[\prod_{\alpha=M+1}^N \mathcal{B}(\lambda_\alpha)\right]q_1\left[\prod_{\alpha=1}^M\mathcal{B}(\lambda_\alpha)\right]w_N^+ \end{aligned} \tag{6}$$

$$f_4^M = Z_N^{-1} w_N^- \left[\prod_{\alpha=M+1}^{N} \mathcal{B}(\lambda_\alpha)\right]\left[\sinh\left(\lambda_M + \frac{1}{2}\eta + \zeta_+\right) B(\lambda_M) q_1 D(-\lambda_M)\right.$$
$$\left. + \sinh\left(\lambda_M + \frac{1}{2}\eta - \zeta_+\right) D(\lambda_M) q_1 B(-\lambda_M)\right]\left[\prod_{\alpha=1}^{M-1} \mathcal{B}(\lambda_\alpha)\right] w_N^+.$$

Here $q_1 = \frac{1}{2}(1 - \sigma_1^3)$, and $p_1 = \frac{1}{2}(1 + \sigma_1^3)$. f_1^M and f_2^M describe the probability that the spin on the first column is turned down just on the $2M$th and $(2M-1)$th rows, respectively. Also, f_3^M and f_4^M describe the probability that the spin on the first column is turned down before the $2M$th and $(2M-1)$th rows, respectively.

3. Determinant representations

To derive the reduction formulae for the correlation functions, we can decompose the bulk monodromy matrix $T(\lambda_\alpha)$ into the matrix product of two monodromy matrices in the αth row:

$$T(\lambda_\alpha) = \hat{T}_\alpha(\lambda_\alpha) L_{\alpha 1}(\lambda_\alpha, \nu_1) \qquad \hat{T}_\alpha(\lambda_\alpha) = L_{\alpha N}(\lambda_\alpha, \nu_N) \cdots L_{\alpha 2}(\lambda_\alpha, \nu_1). \tag{7}$$

Utilizing the expression of $L_{\alpha 1}(\lambda_\alpha - \nu_1)(1)$, f_1^M can be written as

$$f_1^M = \frac{-\sinh\eta}{\sinh\left(\nu_1 + \frac{1}{2}\eta - \lambda_M\right)} \prod_{\alpha=1}^{M} \frac{\left[\sinh^2\left(\nu_1 + \frac{1}{2}\eta\right) - \sinh^2\lambda_\alpha\right]}{\left[\sinh^2\left(\nu_1 - \frac{1}{2}\eta\right) - \sinh^2\lambda_\alpha\right]}$$
$$\times Z_N^{-1} w_{N-1}^- \left[\prod_{\gamma=M+1}^{N} \hat{\mathcal{B}}(\lambda_\gamma)\right] \hat{\mathcal{D}}(\lambda_M) \left[\prod_{\gamma=1}^{M-1} \hat{\mathcal{B}}(\lambda_\gamma)\right] w_{N-1}^+ \tag{8}$$

where $w_{N-1}^+ = \prod_{i=2}^{N} \uparrow_i$. $\hat{\mathcal{B}}(\lambda_\gamma)$ and $\hat{\mathcal{D}}(\lambda_\gamma)$ are the operators acting on columns from 2 to N and are defined as

$$\hat{U}^{t_\alpha}(\lambda_\alpha) = \hat{T}^{t_\alpha}(\lambda_\alpha) K_+(\lambda_\alpha) \sigma_\alpha^2 \hat{T}(-\lambda_\alpha) \sigma_\alpha^2 = \begin{pmatrix} \hat{\mathcal{A}}(\lambda_\alpha) & \hat{\mathcal{C}}(\lambda_\alpha) \\ \hat{\mathcal{B}}(\lambda_\alpha) & \hat{\mathcal{D}}(\lambda_\alpha) \end{pmatrix}_{[\alpha]}.$$

The commutation relations between $\hat{\mathcal{A}}$, $\hat{\mathcal{B}}$, $\hat{\mathcal{C}}$ and $\hat{\mathcal{D}}$ are similar to those between $\mathcal{A}(\lambda_\alpha)$, $\mathcal{B}(\lambda_\alpha)$, $\mathcal{C}(\lambda_\alpha)$ and $\mathcal{D}(\lambda_\alpha)$. Applying these commutation relations (4) repeatedly, we have

$$\hat{\mathcal{A}}(\lambda_\alpha)\left[\prod_{\beta=1}^{\alpha-1} \hat{\mathcal{B}}(\lambda_\beta)\right] w_{N-1}^+$$
$$= \sum_{\beta=1}^{\alpha} \frac{\prod_{i=2}^{N}\left[\sinh^2\left(\nu_i + \frac{1}{2}\eta\right) - \sinh^2\lambda_\beta\right]}{\prod_{\gamma=1,\neq\beta}^{\alpha}(\sinh^2\lambda_\beta - \sinh^2\lambda_\gamma)} a_\beta^{(\alpha)} \left[\prod_{\gamma=1,\neq\beta}^{\alpha} \hat{\mathcal{B}}(\lambda_\gamma)\right] w_{N-1}^+$$
$$\hat{\mathcal{D}}(\lambda_\alpha)\left[\prod_{\beta=1}^{\alpha-1} \hat{\mathcal{B}}(\lambda_\beta)\right] w_{N-1}^+ \tag{9}$$
$$= \sum_{\beta=1}^{\alpha} \frac{\prod_{i=2}^{N}\left[\sinh^2\left(\nu_i + \frac{1}{2}\eta\right) - \sinh^2\lambda_\beta\right]}{\prod_{\gamma=1,\neq\beta}^{\alpha}(\sinh^2\lambda_\beta - \sinh^2\lambda_\gamma)} d_\beta^{(\alpha)} \left[\prod_{\gamma=1,\neq\beta}^{\alpha} \hat{\mathcal{B}}(\lambda_\gamma)\right] w_{N-1}^+.$$

$a_\beta^{(\alpha)}$ and $d_\beta^{(\alpha)}$ are functions depending on parameters α and β,

$$a_\beta^{(\alpha)} = \frac{\sinh(\eta)\sinh(2\lambda_\alpha)\sinh(\eta+2\lambda_\beta)}{\sinh(2\lambda_\beta)}\Big[\sinh(\eta+\lambda_\beta-\lambda_\alpha)\Lambda_\beta^{(\alpha)}(\lambda_\beta) - \sinh(\eta-\lambda_\beta-\lambda_\alpha)\Lambda_\beta^{(\alpha)}(-\lambda_\beta)\Big]$$

$$d_\beta^{(\alpha)} = -\frac{\sinh(\eta)\sinh(\eta+2\lambda_\beta)}{\sinh(2\lambda_\beta)}\Big[\sinh(\lambda_\alpha+\lambda_\beta)\Lambda_\beta^{(\alpha)}(\lambda_\beta) - \sinh(\lambda_\alpha-\lambda_\beta)\Lambda_\beta^{(\alpha)}(-\lambda_\beta)\Big]$$

$$\Lambda_\beta^{(\alpha)}(\lambda_\beta) = \sinh\left(\lambda_\beta+\frac{1}{2}\eta-\zeta_+\right)\frac{\prod_{\gamma=1}^{\alpha-1}[\sinh^2(\lambda_\beta+\eta)-\sinh^2\lambda_\gamma]}{\prod_{i=2}^{N}\left[\sinh^2\nu_i-\sinh^2\left(\frac{1}{2}\eta+\lambda_\beta\right)\right]}.$$

Using equation (9), equation (8) can be written as

$$f_1^M = \frac{-\sinh\eta}{\sinh\left(\nu_1+\frac{1}{2}\eta-\lambda_M\right)}\prod_{\gamma=1}^{M}\frac{\left[\sinh^2\left(\nu_1+\frac{1}{2}\eta\right)-\sinh^2\lambda_\gamma\right]}{\left[\sinh^2\left(\nu_1-\frac{1}{2}\eta\right)-\sinh^2\lambda_\gamma\right]} \times\sum_{\alpha=1}^{M}\frac{\prod_{i=2}^{N}\left[\sinh^2\left(\nu_i+\frac{1}{2}\eta\right)-\sinh^2\lambda_\alpha\right]}{\prod_{\gamma=1,\neq\alpha}^{M}(\sinh^2\lambda_\alpha-\sinh^2\lambda_\gamma)}a_\alpha^{(M)}\frac{Z_{N-1}(\{\lambda_\beta\}_{\beta\neq\alpha},\{\nu_k\}_{k\neq1})}{Z_N(\{\lambda_\beta\}_N,\{\nu_k\}_N)}. \tag{10}$$

As in the case of the periodic boundary condition, the partition function $Z_N(\{\lambda_\beta\}_N,\{\nu_k\}_N)$ in the case of the reflecting boundary condition may also be expressed as a determinant of the usual functions of the spectrum parameters $\{\lambda_\alpha\}_N$ and $\{\nu_k\}_N$ [8]

$$\begin{aligned} Z_N(\{\lambda_\alpha\}_N,\{\nu_k\}_N) &= \frac{\prod_{\alpha=1}^{N}\prod_{i=1}^{N}\left[\sinh^2\left(\nu_i+\frac{1}{2}\eta\right)-\sinh^2\lambda_\alpha\right]}{\prod_{1\leqslant i<j\leqslant N}(\sinh^2\nu_j-\sinh^2\nu_i)\prod_{1\leqslant\alpha<\beta\leqslant N}(\sinh^2\lambda_\alpha-\sinh^2\lambda_\beta)} \\ &\quad\times\det{}_N\,\mathcal{X}(\{\lambda_\alpha\}_N,\{\nu_k\}_N) \\ \mathcal{X}_{\alpha i} &= \frac{-\sinh\eta\sinh(2\lambda_\alpha+\eta)\sinh(\nu_i+\zeta_+)}{\left[\sinh^2\left(\nu_i+\frac{1}{2}\eta\right)-\sinh^2\lambda_\alpha\right]\left[\sinh^2\left(\nu_i-\frac{1}{2}\eta\right)-\sinh^2\lambda_\alpha\right]}. \end{aligned} \tag{11}$$

Substituting equation (11) into equation (10) and taking into account that $\prod_{\gamma=M+1}^{N}(\sinh^2\lambda_\alpha - \sinh^2\lambda_\gamma) = 0$ when $\alpha > M$, f_1^M can be written as

$$\begin{aligned} f_1^M &= \frac{-\sinh\eta\sinh\left(\nu_1+\frac{1}{2}\eta+\lambda_M\right)\prod_{i=2}^{N}(\sinh^2\nu_i-\sinh^2\nu_1)}{\prod_{\gamma=1}^{M}\left[\sinh^2\left(\nu_1-\frac{1}{2}\eta\right)-\sinh^2\lambda_\gamma\right]\prod_{\gamma=M}^{N}\left[\sinh^2\left(\nu_1+\frac{1}{2}\eta\right)-\sinh^2\lambda_\gamma\right]} \\ &\quad\times\frac{\det_N\mathcal{H}_1^M(\{\lambda_\beta\}_{\beta\neq\alpha},\{\nu_k\}_{k\neq1})}{\det_N\mathcal{X}(\{\lambda_\alpha\}_N,\{\nu_k\}_N)} \\ &\quad\begin{cases}\left(\mathcal{H}_1^M\right)_{\alpha i} = \mathcal{X}_{\alpha i} & 2\leqslant i\leqslant N\\ \left(\mathcal{H}_1^M\right)_{\alpha 1} = a_\alpha^{(M)}\prod_{\gamma=M+1}^{N}(\sinh^2\lambda_\alpha-\sinh^2\lambda_\gamma).\end{cases} \end{aligned} \tag{12}$$

Similar to the calculation of f_1^M, f_2^M allows us to express the following determinants:

$$\begin{aligned} f_2^M &= \frac{-\sinh\eta\sinh\left(\nu_1-\frac{1}{2}\eta-\lambda_M\right)\prod_{i=2}^{N}(\sinh^2\nu_i-\sinh^2\nu_1)}{\prod_{\gamma=1}^{M}\left[\sinh^2\left(\nu_1-\frac{1}{2}\eta\right)-\sinh^2\lambda_\gamma\right]\prod_{\gamma=M}^{N}\left[\sinh^2\left(\nu_1+\frac{1}{2}\eta\right)-\sinh^2\lambda_\gamma\right]} \\ &\quad\times\frac{\det_N\mathcal{H}_2^M(\{\lambda_\beta\}_{\beta\neq\alpha},\{\nu_k\}_{k\neq1})}{\det_N\mathcal{X}(\{\lambda_\alpha\}_N,\{\nu_k\}_N)} \\ &\quad\begin{cases}\left(\mathcal{H}_2^M\right)_{\alpha i} = \mathcal{X}_{\alpha i} & 2\leqslant i\leqslant N\\ \left(\mathcal{H}_2^M\right)_{\alpha 1} = d_\alpha^{(M)}\prod_{\gamma=M+1}^{N}(\sinh^2\lambda_\alpha-\sinh^2\lambda_\gamma).\end{cases} \end{aligned} \tag{13}$$

Now we consider functions f_3^M and f_4^M. First, applying equation (7) to equation (6), we can express f_3^M as

$$f_3^M = Z_N^{-1} \sum_{\alpha=1}^{M} \frac{-\sinh\eta \prod_{\beta=1}^{\alpha-1}\left[\sinh^2\left(\nu_1+\frac{1}{2}\eta\right)-\sinh^2\lambda_\beta\right]}{\prod_{\beta=1}^{\alpha}\left[\sinh^2\left(\nu_1-\frac{1}{2}\eta\right)-\sinh^2\lambda_\beta\right]} w_{N-1}^{-} \times \left[\prod_{\gamma=\alpha+1}^{N}\hat{\mathcal{B}}\left(\lambda_\gamma\right)\right]\left\{\left[\sinh\left(\nu_1+\frac{1}{2}\eta+\lambda_\alpha\right)\hat{\mathcal{A}}(\lambda_\alpha) + \sinh\left(\nu_1-\frac{1}{2}\eta-\lambda_\alpha\right)\hat{\mathcal{D}}(\lambda_\alpha)\right]\right\}\left[\prod_{\gamma=1}^{\alpha-1}\hat{\mathcal{B}}(\lambda_\gamma)\right] w_{N-1}^{+}.$$

Secondly, picking the term $\alpha = M$ in the sum and utilizing the relations (9), we have

$$f_3^M = -\sinh\eta\left[\prod_{\beta=1}^{M}\frac{\sinh^2\left(\nu_1+\frac{1}{2}\eta\right)-\sinh^2\lambda}{\sinh^2\left(\nu_1-\frac{1}{2}\eta\right)-\sinh^2\lambda_\beta}\right]\frac{\prod_{i=2}^{N}\left[\sinh^2\left(\nu_i+\frac{1}{2}\eta\right)-\sinh^2\lambda_M\right]}{\prod_{\gamma=1}^{M-1}(\sinh^2\lambda_M-\sinh^2\lambda_\gamma)} \times\left[\frac{a_M^{(M)}}{\sinh\left(\nu_1+\frac{1}{2}\eta-\lambda_M\right)}+\frac{\sinh\left(\nu_1-\frac{1}{2}\eta-\lambda_M\right)d_M^{(M)}}{\sinh^2\left(\nu_1+\frac{1}{2}\eta\right)-\sinh^2\lambda_M}\right] \times\frac{Z_{N-1}(\{\lambda_\beta\}_{\beta\neq M},\{\nu_k\}_{k\neq 1})}{Z_N(\{\lambda_\beta\}_N,\{\nu_k\}_N)}+\cdots. \tag{14}$$

The other terms in equation (14) are terms with scalar products involving the operators $\hat{B}(\lambda_M)$. Because $\mathcal{B}(\lambda_\alpha)$ commutate each other, M spectrum parameters $\lambda_1, \ldots, \lambda_M$ are completely symmetric in f_3^M. It follows that the whole expression in equation (14) is the sum over the cyclic permutations of the elements in the set $\{\lambda_1 \cdots \lambda_M\}_M$,

$$f_3^M = -\sinh\eta\left[\prod_{\beta=1}^{M}\frac{\sinh^2\left(\nu_1+\frac{1}{2}\eta\right)-\sinh^2\lambda_\beta}{\sinh^2\left(\nu_1-\frac{1}{2}\eta\right)-\sinh^2\lambda_\beta}\right]\sum_{\alpha=1}^{M}\frac{\prod_{i=2}^{N}\left[\sinh^2\left(\nu_i+\frac{1}{2}\eta\right)-\sinh^2\lambda_\alpha\right]}{\prod_{\gamma=1,\neq\alpha}^{M}(\sinh^2\lambda_\alpha-\sinh^2\lambda_\gamma)} \times\left[\frac{a_\alpha^{(M)}}{\sinh\left(\nu_1+\frac{1}{2}\eta-\lambda_\alpha\right)}+\frac{\sinh\left(\nu_1-\frac{1}{2}\eta-\lambda_M\right)d_\alpha^{(M)}}{\sinh^2\left(\nu_1+\frac{1}{2}\eta\right)-\sinh^2\lambda_\alpha}\right]\frac{Z_{N-1}(\{\lambda_\beta\}_{\beta\neq\alpha},\{\nu_k\}_{k\neq 1})}{Z_N(\{\lambda_\beta\}_N,\{\nu_k\}_N)}.$$

This reduction formula for f_3^M is similar to that for f_1^M (equation (10)), so we can express f_3^M as

$$f_3^M = \frac{-\sinh\eta\prod_{i=2}^{N}(\sinh^2\nu_i-\sinh^2\nu_1)}{\prod_{\gamma=1}^{M}\left[\sinh^2\left(\nu_1-\frac{1}{2}\eta\right)-\sinh^2\lambda_\gamma\right]\prod_{\gamma=M+1}^{N}\left[\sinh^2\left(\nu_1+\frac{1}{2}\eta\right)-\sinh^2\lambda_\gamma\right]} \times\frac{\det_N\mathcal{H}_3^M(\{\lambda_\beta\}_{\beta\neq\alpha},\{\nu_k\}_{k\neq 1})}{\det_N\mathcal{X}(\{\lambda_\alpha\}_N,\{\nu_k\}_N)} \tag{15}$$

$$\begin{cases}\left(\mathcal{H}_3^M\right)_{\alpha i} = \mathcal{X}_{\alpha i} \qquad 2\leqslant i\leqslant N\\ \left(\mathcal{H}_3^M\right)_{\alpha 1} = \left[\frac{a_\alpha^{(M)}}{\sinh\left(\nu_1+\frac{1}{2}\eta-\lambda_\alpha\right)}+\frac{\sinh\left(\nu_1-\frac{1}{2}\eta-\lambda_M\right)d_\alpha^{(M)}}{\sinh^2\left(\nu_1+\frac{1}{2}\eta\right)-\sinh^2\lambda_\alpha}\right] \\ \qquad\qquad \times\prod_{\gamma=M+1}^{N}(\sinh^2\lambda_\alpha-\sinh^2\lambda_\gamma).\end{cases}$$

Because of $1 = q_1 + p_1$, f_4^M can be expressed through f_2^α and f_3^α as

$$f_4^M = f_3^{M-1} + f_2^M.$$

From equations (13) and (15) and the sum formulae of two determinants, we obtain

$$f_4^M = \frac{-\sinh\eta\sinh\left(\nu_1 - \frac{1}{2}\eta - \lambda_M\right)\prod_{i=2}^N(\sinh^2\nu_i - \sinh^2\nu_1)}{\prod_{\gamma=1}^M\left[\sinh^2\left(\nu_1 - \frac{1}{2}\eta\right) - \sinh^2\lambda_\gamma\right]\prod_{\gamma=M}^N\left[\sinh^2\left(\nu_1 + \frac{1}{2}\eta\right) - \sinh^2\lambda_\gamma\right]} \times \frac{\det_N\mathcal{H}_4^M(\{\lambda_\beta\}_{\beta\neq\alpha}, \{\nu_k\}_{k\neq 1})}{\det_N\mathcal{X}(\{\lambda_\alpha\}_N, \{\nu_k\}_N)} \tag{16}$$

$$\begin{cases} \left(\mathcal{H}_4^M\right)_{\alpha i} = \mathcal{X}_{\alpha i} \qquad 2 \leqslant i \leqslant N \\ \left(\mathcal{H}_4^M\right)_{\alpha 1} = d_\alpha^{(M)}\prod_{\gamma=M+1}^N(\sinh^2\lambda_\alpha - \sinh^2\lambda_\gamma) + \sinh\left(\nu_1 - \frac{1}{2}\eta + \lambda_M\right) \\ \qquad \times\left[\dfrac{a_\alpha^{(M-1)}}{\sinh\left(\nu_1 + \frac{1}{2}\eta - \lambda_\alpha\right)} + \dfrac{\sinh\left(\nu_1 - \frac{1}{2}\eta - \lambda_M\right)d_\alpha^{(M-1)}}{\sinh^2\left(\nu_1 + \frac{1}{2}\eta\right) - \sinh^2\lambda_\alpha}\right] \\ \qquad \times\prod_{\gamma=M}^N(\sinh^2\lambda_\alpha - \sinh^2\lambda_\gamma). \end{cases}$$

4. Discussion

The calculations of the partition functions and the correlation functions are two main problems in exactly solved statistical mechanics. In this paper, we have calculated four types of correlation functions which are related to the boundary spontaneous polarization for the six-vertex model with a reflecting boundary condition. These results allow determinant representations and generalize the known result for the partition function. Because of the existence of the reflecting boundary, it is difficult to calculate the spontaneous polarization at an arbitrary point of the lattice. So it is interesting to investigate a new approach for the calculation of arbitrary correlation functions.

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